Last time: $p \in C_q(k,n)$ iff there are f.d. Hilbert spaces H_A, H_B, POVMs $(A^x)_{x\in[k]}$, $(B^y)_{y\in[k]}$ on H_A, H_B resp. ($A^x = (A^x_a)_{a\in[n]}$ POVM on H_A) and state (unit vector) $\xi \in H_A \otimes H_B$ so that $p(a,b|x,y) = \langle (A^x_a \otimes B^y_b)\xi, \xi\rangle$

$$\mathrm{val}^*(\mathcal{G}) := \sup_{p\in C_q(k,n)} \mathrm{val}(\mathcal{G}, p)$$

Def A language L is in MIP* if there is an efficient mapping $z \mapsto \mathcal{G}_z$ s.t.
- If $z \in L$, then $\mathrm{val}^*(\mathcal{G}_z) \geq \frac{2}{3}$
- If $z \notin L$, then $\mathrm{val}^*(\mathcal{G}_z) \leq \frac{1}{3}$.

MIP vs. MIP*?
Not obvious.

Last time: Showed there is a game $\mathcal{G}$ s.t. $\mathrm{val}(\mathcal{G}) < \mathrm{val}^*(\mathcal{G})$.

If $L \in \mathrm{MIP}$ via $z \mapsto \mathcal{G}_z$ and $z \notin L$, then $\mathrm{val}(\mathcal{G}_z) \le \frac{1}{3}$ but $\mathrm{val}^*(\mathcal{G}_z) > \frac{1}{3}$ possible.

Nevertheless:

Thm (Ito & Vidick) $\mathrm{MIP} \subseteq \mathrm{MIP}^*$.

Key: $\mathrm{NEXP} \subseteq \mathrm{MIP}^*$

$$\begin{array}{c}\mathrm{NEXP} \subseteq \mathrm{MIP}^* \\ \parallel \\ \mathrm{MIP}\end{array}$$

Thm (Natarajan & Wright)

$$\begin{array}{c}\mathrm{NEEXP} \subseteq \mathrm{MIP}^* \\ \not\cup \\ \mathrm{NEXP} = \mathrm{MIP}\end{array}$$

Silly: $MIP^* \subseteq RE$ = set of recursively enumerable languages.

Why? If $z \mapsto y_z$ witnesses that L belongs to MIP^*, we can eventually know if $z \in L$ when z does belong to L:

Start "computing" $val(y_z, p)$ as p ranges over a computable dense subset of $C_q(k,n)$.

Key: States on $M_k(\mathbb{C}) = \mathcal{B}(k\text{-dim})$ are of the form $Tr(\cdot\, a)$ where $a \geq 0$, $Tr(a) = 1$.

If $val(y_z) > \frac{1}{3}$, you'll eventually see that.

Thm $MIP^* = RE$! More precisely:
there is an effective mapping
$\underline{M} \mapsto \mathcal{G}_{\underline{M}}$ (Turing machines to games)
s.t. (on empty input)
- If $\underline{M}$ halts, then $val^*(\mathcal{G}_{\underline{M}}) = 1$.
- If $\underline{M}$ doesn't halt, then $val^*(\mathcal{G}_{\underline{M}}) \le \frac{1}{2}$.

Traditional Route from $MIP^* = RE \Rightarrow \neg QWEP$

Step 1 $MIP^* = RE \Rightarrow \neg$ Tsirelson's Problem

$C_{qs}(k,n)$ = same as $C_q(k,n)$ but allow ∞-dim. Hilbert spaces

$\uparrow$

quantum spatial

- $C_q(k,n) \subseteq C_{qs}(k,n)$
- WLOG, assume Hilb. spaces are separable
- $C_{qs}(k,n) \subseteq C_q(k,n)$

??

- $C_{qs} \subseteq [0,1]^{k^2n^2}$ convex

Another model (coming from quantum field theory):

possibly ∞-dim

- Single Hilbert space H, state $\xi \in H$
- Alice & Bob have POVMs $(A^x)_{x\in[k]}$, $(B^y)_{y\in[k]}$ on H.
- Want these measurements to be able to be made simultaneously!

Algebraically: $A^x_a B^y_b = B^y_b A^x_a \;\; \forall x,y,a,b.$

$$p(a,b|x,y) = \langle A^x_a B^y_b \xi, \xi \rangle$$

quantum commuting strategies $C_{qc}(k,n)$

Clear: $C_{qs}(k,n) \subseteq C_{qc}(k,n)$.

$A^x_a \otimes I_{H_B}$ commutes with $I_{H_A} \otimes B^y_b$.

acting on $H = H_A \otimes H_B$

closed

Tsirelson: Claimed $C_{qs}(k,n) = C_{qc}(k,n)$.
Issue w/ proof. Wasn't even clear if $C_{qs}(k,n)$ closed.

$$C_{qa}(k,n) := \overline{C_{qs}(k,n)} = \overline{C_q(k,n)} \subseteq C_{qc}(k,n)$$

<u>Tsirelson's Problem</u> $C_{qa}(k,n) = C_{qc}(k,n)$?

$$C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc}$$

closed

<u>Slofstra ('19)</u>

- $C_{qs} \neq C_{qc}$
- $C_{qs} \neq C_{qa}$

commuting value

Def $\mathrm{val}^{co}(\mathcal{G}) := \sup_{p \in C_{qc}(k,n)} \mathrm{val}(\mathcal{G}, p)$

$$\mathrm{val}(\mathcal{G}) \leq \underline{\mathrm{val}^*(\mathcal{G})} \leq \mathrm{val}^{co}(\mathcal{G})$$

Tsirelson's Problem $\Leftrightarrow$ $\mathrm{val}^*(\mathcal{G}) = \mathrm{val}^{co}(\mathcal{G})$ $\forall \mathcal{G}$.

<u>Fact</u>: $\mathrm{val}^{co}(\mathcal{G})$ can always be effectively approximated from above.

- $\exists$ f.p. group $G_{\mathcal{G}}$ (depends on k & n) and $\eta_{\mathcal{G}} \in C^*(G_{\mathcal{G}})$ s.t.

$$\mathrm{val}^{co}(\mathcal{G}) = \|\eta_{\mathcal{G}}\|.$$

- Thm of Fritz, Netzer, & Thom: for any f.p. group G, $\|\cdot\|$ on $C^*(G)$

can be effectively approximated from above (semidefinite programming)

Saw: $\mathrm{val}^*(\mathcal{G})$ can be effectively approx. from below.

<u>So</u> Tsirelson $\Longrightarrow$ $\mathrm{val}^*(\mathcal{G}) = \mathrm{val}^{co}(\mathcal{G})$ is computable

$\Longrightarrow$ every language in MIP^* is decidable!

$\not\downarrow$ to $\mathrm{MIP}^* = \mathrm{RE}$!

<u>Remark</u> $\mathrm{val}^{co}(\mathcal{G})$ eff app from above

$\Longrightarrow$ $\mathrm{MIP}^{co} \subseteq \mathrm{coRE}$ = languages whose complement is RE

Open problem: $\mathrm{MIP}^{co} = \mathrm{coRE}$?

To finish: QWEP $\Longrightarrow$ Tsirelson

$\mathbb{C}^n$: abelian C^*-algebra $= C(n \text{ pointspace})$
e_a $a^{\underline{th}}$ standard basis vector, $a = 1, \dots, n$.

<u>Key Observation 1</u> There is a 1-1 correspondence between POVMs $\{A_1, \dots, A_n\}$ on H and ucp maps $\Phi : \mathbb{C}^n \to B(H)$ via $\Phi(e_a) = A_a$.

<u>Key Observation 2</u> There is a 1-1 correspondence between families of POVMs $A^x = (A^x_1, \dots, A^x_n)$, $x \in [k]$, on $B(H)$ and ucp maps

$$\Phi : \underbrace{\mathbb{C}^n * \cdots * \mathbb{C}^n}_{} \longrightarrow B(H)$$

k times

given by $\Phi(e_a^x) = A_a^x$

e_a^x is the version of e_a in the $x^{\underline{th}}$ copy of $\mathbb{C}^n$.

- Uses a result of Florin Boca that the maps in Observation #1 can be "glued together" to give ucp map.

$\mathbb{C}^n \cong C^*(\mathbb{Z}_n)$

If $\mathbb{Z}_n = \langle u \rangle$, then map u to

$\sum \exp(\frac{2\pi i}{a}) e_a \in \mathbb{C}^n$.

"discrete Fourier transform"

$$\mathop{\ast}_{k=1}^{n} \mathbb{C}^n \cong \mathop{\ast}_{k=1}^{n} C^*(\mathbb{Z}_n) \cong C^*(\underbrace{\mathop{\ast}_{k=1}^{n} \mathbb{Z}_n}_{\mathbb{F}(k,n)})$$

Recap Families (A^x) of POVMs correspond to ucp maps
$\Phi : C^*(\mathbb{F}(k,n)) \longrightarrow B(H)$

Thm For $p \in [0,1]^{k^2 n^2}$, we have:

① $p \in C_{qa}(k,n)$ iff there is a state ϕ on $C^*(\mathbb{F}(k,n)) \otimes_{min} C^*(\mathbb{F}(k,n))$ for which $p(a,b|x,y) = \phi(\underbrace{e^x_a \otimes e^y_b}_{\text{lives in } \odot})$.

positive, linear function, $\phi(1)=1$.

② Same for C_{qc} using $\otimes_{max}$.

If $\xi \in H$, get <u>vector state</u> ω_ξ on $B(H)$: $\omega_\xi(a) := \langle a\xi, \xi \rangle$